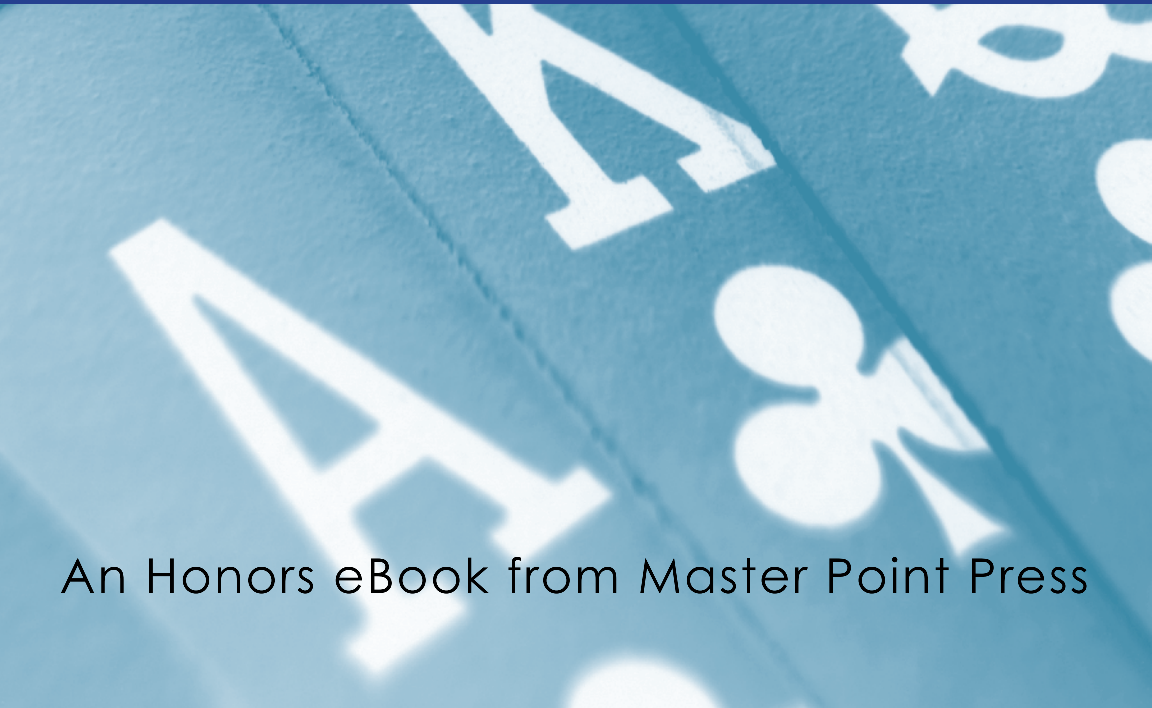
Émile Borel & André Chéron

THE MATHEMATICAL THEORY OF BRIDGE

TRANSLATED BY ALEC TRAUB

REVISED AND CORRECTED BY GILES LAURÉN



An Honors eBook from Master Point Press

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Master Point Press
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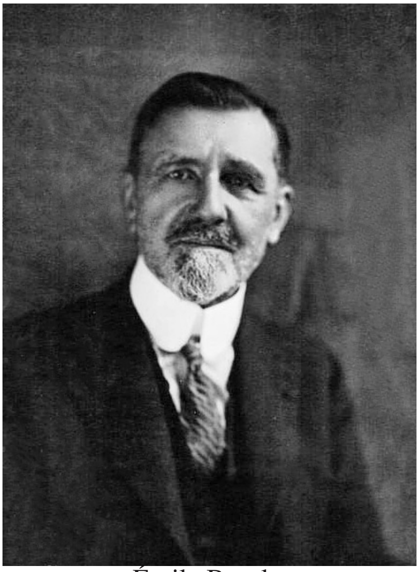
www.masterpointpress.com
www.bridgeblogging.com
www.teachbridge.com
www.ebooksbridge.com

ISBN: 978-1-55494-586-3

Cover Design: Olena S. Sullivan/New Mediatrix

1 2 3 4 5 6 20 19 18 17


Émile Borel


André Chéron

Dedication.

This revised edition of *The Mathematics of Bridge* is dedicated to Judith Skellenger who introduced me, and generations of others, to the challenges and pleasures of this delightful game. Her students today gain from the wisdom and experience of her second husband, Jerry Skellenger.

And to Martha Russo, the most complaisant Club President and Director any enthusiast could have; ever patient, she encouraged me to find my own game, experiment, and made germane and pertinent suggestions.

And finally, to all of my friends at the Saint Augustine Duplicate Bridge Club, whose good-will and tolerance are heroic - despite their sartorial denial.

Foreword to New Edition.

The concept and application of mathematics to games of chance dates *grosso modo* from Blaise Pascal [1623-62] and John Law [1671-1729].

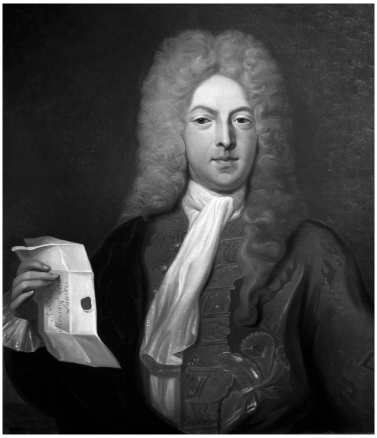
The first half of the 20th century was fecund in first-water mathematicians: Poincaré, Hadamard, Wiener, Russell, Whitehead, Hardy, Ramanujan, Heisenberg, Schroedinger, Von Neumann, Erdos and others. Émile Borel (1871-1956) belonged to this generation and made contributions to mathematics perhaps more pervasive (though underestimated) than his better known contemporaries. He seems to have been the first to return to probability theory and its applications to games of chance. He published more than fifty papers on probability between 1905 and 1950 and between 1921 and 1927 he published papers on game theory and was first to define games of strategy; publishing various papers and finally, *Applications aux jeux de hasard* in 1938 of which this present is a part. It is likely that Von Neumann's interest and 1928 paper: *Zur Theorie der Gesellschaftsspiele* was sparked by Borel's work. Von Neumann's work with game theory was later the basis for his seminal collaboration with Morgenstern [1944]: *Theory of Games and Economic Behaviour* which was in turn quickly adopted by such influential economists as Samuelson and thence by the business schools. It can therefore be argued that Borel introduced our era of probabilistic, quantitative, decision making, so adaptable to the computer and thereby so pervasive today. Indeed, McGeorge Bundy, the Edsel, the Vietnam body counts, learning algorithms, investment theory, portfolio insurance, programme trading, and A.I. all flow from Borel.

Apart from his chair at the École normale supérieure, Borel was active in politics as a member of the National Assembly and Minister of Marine. He was later decorated for his service to the Resistance while in his 70's.

While ongoing experience with statistical reasoning, *i.e*. Artificial Intelligence, is producing Black Swans and 'unintended consequences', it continues to be perfectly apt for the clean numbers of finite games of chance, such as Bridge, and it is hoped that the reintroduction of this book may prove useful to reflective bridge players.

André Chéron (1895-1980) was a top ranked chess player of the generation of Capablanca, Lasker & Alekhine; they all turned to bridge as a 'more interesting game' in the 1930's.

This new edition corrects numerical errors found in both earlier texts; it revises the previous English translation where needed and corrects a number of textual and typographical errors in the 1974 edition. It is hoped the new typology will improve clarity and that the re-inclusion of the Tables in the text, as in the original edition, will be a convenience. The chapter and tables on Contract & Plafond scoring have been retained to demonstrate the variables that determine the best-line-of-play. The chapters on shuffling, although no longer applicable to Duplicate Bridge, have been retained for the benefit of those interested in the mathematics of *all* card games. All, it is hoped, without too many new errors being introduced.



John Law

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PREFACE.

The interest which bridge experts have shown in some pages of one of the parts of my Treatise on probabilities[1] has encouraged me to pursue the study of this game, which daily becomes more popular, by using all the resources of theory and calculation. Such a study is not only of interest to players; it also contributes to the progress of the science of probabilities, which owes so much to the study of games of pure chance and of which many of the applications are valuable in the theory of games in which both chance and psychology apply, games of which bridge is without doubt the one in greatest favour. This work therefore gives numerous examples of precautions necessary to avoid certain errors in applying probabilities.

It would have been impossible for me to undertake such a task if I had not the collaboration of a bridge expert, for theory cannot supplant the science of the game, and this is only acquired by lengthy experience and daily consideration of such experience. I had the good fortune to meet M. André Chéron who for some years had been in charge of the bridge columns in "Temps" and "L'Illustration". M, André Chéron is not only one of the most distinguished bridge experts, but he has a lively feeling for numerical calculations, this feeling being accompanied by exceptional ability in handling and interpreting such calculations. He therefore not only gave me very valuable advice, but also the results of a large number of unpublished calculations, of which we find the most important in this work in the form of Tables and which will now constitute the essential foundations for any serious study of bridge. The data which he accumulated patiently over the years and which he has classified enabled him to reply

1 *Traité du calcul des probabilités et de ses applications*, Tome IV, fascicule II; Application aux jeux de hasard, edited by Jean Ville, based on a course by Émile Borel.

almost immediately to the most important questions regarding probabilities in bridge which could be put to him.

There are many books on bridge and, in almost every case, the authors use, from time to time, a few results of the elementary calculus of probabilities. But there are certain problems which are never approached and there are others where the solutions given from time to time are not accurate, mainly because the authors have relied upon remarks regarding the variation in probability during the course of play, remarks which I have corrected in my *Traité du calcul des probabilitiés* (Tome IV Fasc. II).

This is not a work on bridge. We seldom touch on the rules of the game, and we also assume that the reader is familiar with the standard practice in bidding, finessing, signals, squeeze play, &c. We also assume that the reader has an elementary knowledge of the theory of probabilities, a knowledge which anyone who has studied can acquire in a few days from any elementary précis of the calculus of probabilities.

We intend to provide our readers simultaneously with a method and a large number of numerical results which will facilitate the application of this method to each concrete case which can arise, cases which cannot all be studied since they are innumerable, even if we arrange them in vast categories.

All bridge players who have managed, either through reflection or calculation, to establish rules of conduct in certain delicate situations will generally find confirmation here of such rules. Further, in most cases this confirmation will state the probability of success of each rule, something which is most important. If one method of winning a decisive trick succeeds 60 times in 100 and a second method 40 times in 100 it is obviously preferable to use the first, but the difference is nevertheless sufficiently small for a good player to use the second if the circumstances peculiar to the hand (bidding, method of play by partner or opponent, state of the score) gives an indication, even vague, in its favour. If, on the contrary, the first method succeeds in 90 cases out of 100 and the second only in 10 the latter will only be used if the peculiarities of the hand indicate almost with certainty that it will succeed.

Sometimes it will happen that our studies contradict certain rules which a player has formed. When this happens he will be led to reflect, if necessary, to check our calculations and arguments and, if he doesn't find a mistake, to modify his technique in the light of fresh knowledge.

In certain cases, always clearly stated, calculations are preceded by hypotheses about the players' psychology and about their methods of bidding and play. It is obvious that these hypotheses are not as precise as calculations. Any alert player can dispute them freely, whereas the results of calculations are not open to dispute – they are either correct or false.

As M. André Chéron mentions in his Remarks which follow, we have every reason to believe that we have, unless accidentally, avoided all calculating and printing errors.

Summing up, if this book is not a bridge book, it does provide useful information to all who know the game and who have not a dread of numbers and calculations, information which is not contained in any bridge work.

On the other hand, those mathematicians who have some slight knowledge of bridge will find here an example of methods which can be used for the study of many other probability problems. It is for this reason that this book is placed in our collection of monographs.

It is necessary to note that the theories and calculations which appear in this work are, with the exception of some examples, entirely independent of the particular laws governing the game of bridge. They are equally applicable to games analogous to bridge which are played with 52 cards, such as whist or boston. The

Théorie mathematique du bridge is thus a basic work, independent of the material it uses.

Paris, May 1939 . Émile BOREL.

P.S.

This work was ready in May, 1939. The proofs were to be corrected during the vacation and it was due to appear in October. The war resulted in serious delay and did not allow me to devote to the proof corrections the care which I would have given them in peacetime. It is my pleasant duty to thank M. André Chéron for the weighty task which he therefore had to undertake almost unaided under particularly difficult circumstances. Not only did he correct the proofs meticulously and verify all the calculations; he also made additions and commentaries for the purpose of elucidating delicate questions for those readers who are bridge players rather than mathematicians and also gave, in all their applications, the results of our arguments and our calculations.

Thus bridge players have been presented not only with a complete practical collection of the most useful probabilities with detailed commentaries on their exact application, but also with a research instrument which will enable them to solve innumerable problems which can arise and which cannot all find space in this book.

Paris, February, 1940. Émile BOREL.

The authors wish to thank Gauthier-Villars, their publishers, for the care which they have taken in composition and correction, particularly of the Tables. They also express their gratitude to the Directors of Gauthier-Villars for completing this enterprise in spite of the difficulties of all kinds caused by the war.

1 March, 1940. Émile BOREL, André CHÉRON.

PREFACE TO THE SECOND EDITION.

This second edition differs from the first on the following points:

Correction of a few material errors.

The addition of an entirely new chapter which, we hope, will be of particular interest to bridge players: Practical Applications of Bayes' Theorem, in which they will find the solution to concrete and subtle problems of play which arise at the bridge table.

1954. The authors.

Remarks concerning the calculations in this work.

All the probabilities and all the figures quoted in this work have been carefully calculated by us with the aid of a calculating machine but without using Stirling's Formula or logarithms, which only give approximate answers.

All the probabilities and all the figures used for the calculation of probabilities and the figures quoted in this work have also been calculated by us in accordance with the method already stated. We have therefore an almost complete Pascal's Arithmetical Triangle from C_4^2 to C_{52}^{26} .

This work does not contain any figure, of whatever nature, taken from another source. We have made an absolute rule, admitting no exceptions, to calculate or recalculate everything ourselves.

To know whether certain long and complicated or short and simple calculations were made for the first time by a specified person in a definite year is certainly not without interest or importance. The date of publication is decisive. Let us leave the mathematical historian to his task and keep to ours, which is to give rigorously correct and controlled figures and theories.

There is no doubt that others have already calculated a certain number of the probabilities we now publish. We do not have the monopoly of calculating the probabilities applicable to bridge. *But we have not taken any notice of these works and have not copied any part of them*. Is it because we consider them valueless? Not in the least. On the contrary, we regard them as most valuable. If the calculations agree with ours one more proof, and not the least valuable, is added to the proofs already existing of the accuracy of our calculations. If there is an inadmissible discrepancy between our calculations and those of others this is proof that someone has erred. It only remains to make the necessary verification to the greater profit of the Science of bridge. The exact and precise localisation of the discrepancy will reduce the work of verification considerably. If there is a slight discrepancy, for example in the third or fourth decimal place, there is the presumption that the discrepancy is due solely *to the difference in precision in the calculating methods used*. Those who use approximation methods, (Stirling's formula, logarithms, slide rules)m, by doing so renounce the absolute precision which is the prerogative solely of our method. By this we do not intend to assert the infallibility of the worker who uses a calculating machine. We simply mention the obvious superiority of this method over all others, as far as precision is concerned.

What are the reasons entitling us to believe in the accuracy of our calculations until the contrary is proved?

First of all there is the use of the calculating machine, which reduced to a minimum the chances of a mechanical error. These chances are not eradicated completely as may be believed by those who have not had experience of such machines. Naturally, when an error occurs it is *never* the fault of the machine but is *always* that of the user. The perfected machine we used makes such an error difficult. In fact it incorporates mechanical transmission of subtotals. Let us assume that we want to obtain the product of 999 x 999 x 999. 999 x 999 = 998 001. If we are using a machine without mechanical transmission the product 998 001 must be noted, the first risk of error. Then the operator must set it on the machine to be multiplied by 999 again.This registration is a second source of error. The chance of error is proportional to the number of operations performed by an operator. On a machine with mechanical transmission the product 998 001 is set automatically in the operating register ready to be multiplied again by 999. And this operation can be repeated until the capacity of the machine is reached. The capacity of our machine is 18 digits in the product register, 12 digits for original entry and 10 digits for multiplier or divisor. These repeated multiplications are the daily currency of our factorial calculations. We note that, owing to decimal linkage, the product 998 001 is obtained with two turns of the handle and two movements of the carriage instead of 27 turns of the handle and three movements of the carriage required with a machine which has not got such linkage.[2]

This is our first reason for claiming accuracy.

Here is our second, taking Table **3** as an example.

We have among our personal papers a similar table for the 39 hands worked out up to and including the seventh of the decimal places, going even further for suits with nine or more cards. As

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2 For those readers unacquainted with manual machines used in the 1930's, the type used by the authors enabled them to multiply by 999 by placing the carriage in the thousands position and rotating the handle one turn clockwise. The carriage was then returned to the units position and the handle rotated once anticlockwise. With less sophisticated machines it was necessary to turn the handle nine times in the unit position, nine times in the tens position and nine also in the hundreds position.

we consistently state in this work, the last decimal is given rounded up *when the following decimal (that is to say the first decimal not printed*) is a 6, 7, 8, 9 or & 5 which is not itself rounded up. In the opposite case the last decimal given is never rounded up. In adding the 39 probabilities in our personal papers our total is 100,000 000 2%.

In adding the 39 probabilities of Table **3** in this work (we disregard the sixth decimal when there is one, and we round up the fifth decimal as explained above when necessary) our total is 100,000 03%.

The fact that in both cases the total is above 100% when in theory it should be exactly 100% is no proof of error. Such a slight variation, either upwards or downwards, is absolutely normal. The last decimal place is approximate, either upwards or downwards, and the sum of the amounts added to round up the fifth decimal cannot compensate exactly for the sum of the quantities disregarded. We could have followed certain writers in falsifying three probabilities by deducting 0,000 01% from each, thus totalling exactly 100%, but we condemn this practice completely and have never followed it.

We direct the reader's attention to another point.

Let us take Table **23** as an example. It gives the partial probability of 4-3-2 as 7,527%. As 4-3-2 has 6 permutations the total probability of 4-3-2 would seem to be:

7,527% **x** 6 = 45,162% – but Table **22** shows it as 45,160%.

The reader should avoid the hasty conclusion that this is an error. The two probabilities are correct, and in each the third decimal is approximate. Here is the explanation of this common occurrence.

The partial probability of 4-3-2 is

$$\frac{22{,}194\,121\,\%\ \text{x}\ 33{,}913\,043\,\%}{100} = 7{,}5267\%$$

The total probability of 4-3-2 is

$$7{,}526\ 7\% \times 6 = 45{,}1602\%,$$

so that the 45,160% given in Table **22** is equally correct. If, in order to preserve the relationship between the two tables, we had given 45,162% we would have given too high a probability and committed a definite error.

All the tables printed in this book were calculated to a number of decimal places greater than those printed. For example, we have "Dummy Expectancy" tables in our possession which are correct to six places of decimals. It seemed to us sufficient to give only four here.

Finally, a last method of verification consisted in calculating the *same probability by two (and sometimes three and four) different methods*. Here is an example. The probability of a deal in which none of the four players has a singleton or a void (which we call an "*ACCIDENT*") is 20,628 055% calculated by the method of coefficients, which is by far the quickest, the simplest and the most certain. We have calculated this same probability by an *entirely different* method and our result was 20,628 055%. This is what we did.

The combined probability that we have a 4-4-3-2 hand and that the other three hands contain no *ACCIDENT* is:

$$\frac{21{,}551\,175\,6\,\%\ \text{x}\ 34{,}648\,039\,\%}{100} = 7{,}467\ 059\ 73\%.$$

The combined probability that we have a 4-3-3-3 hand and that the other three hands contain no *ACCIDENT* is:

$$\frac{10{,}536\,1303\,\%\ \text{x}\ 36{,}780\,996\,\%}{100} = 3{,}875\ 293\ 66\%.$$

The combined probability that we have a 5-3-3-2 hand and that the other three hands contain no *accident* is:

$$\frac{15{,}516\,846\,5\,\%\,\text{x}\,31{,}321\,956\,\%}{100} = 4{,}860\ 179\ 83\%.$$

The combined probability that we have a 5-4-2-2 hand and that the other three hands contain no *accident* is:

$$\frac{10{,}579\,668\,0\,\%\,\text{x}\,29{,}507\,462\%}{100} = 3{,}121\ 791\ 51\%.$$

The combined probability that we have a 6-3-2-2 hand and that the other three hands contain no *accident* is:

$$\frac{5{,}642\,489\,62\,\%\,\text{x}\,22{,}167\,196\,\%}{100} = 1{,}250\ 781\ 73\%.$$

Finally, the combined probability that we have a 7-2-2-2 hand and that the other three hands contain no *accident* is:

$$\frac{0{,}512\,953\,602\,\%\,\text{x}\,10{,}322\,294\,\%}{100} = 0{,}052\ 948\ 58\%.$$

The probability we seek is the sum of these six combined probabilities, that is 20,628 055%. It is a practical certainty, in view of the remarkable coincidence obtained by the two methods, that the probability in question is 20,628 055% and this is the probability we give in Table **9**.

Finally, let us state again that extreme care was taken in correcting the proofs, my wife reading the manuscript and I following the proofs, digit by digit and word for word.

If, in spite of all these precautions and the immense amount of work a fault has slipped into the present work the reader will surely forgive us. *Errare humanum est*.

Leysin, 17th. July, 1939. André CHÉRON.

Decimals or Fractions.

Fractions have the advantage that they give rigorously exact values for the probabilities, but they must be reduced to a common denominator before they can be compared or added, operations which frequently arise. Typographically they also take up more space than decimals. We have therefore sacrificed the superfluous absolute precision of fractions to the convenience of decimals.

But decimal probabilities being only approximate, do we not run the risk that when we add or multiply them, the errors become cumulative in the answer? In practice *we overcome this danger by taking care, as we have done in this work, always to take the result* to at least one decimal *place less than each of the decimal probabilities which have been used to obtain it*.

Let "**e**" be the maximum error in the probabilities A and B. The sum A + B will give us a maximum error of 2**e** and the product A *x* B a maximum error of **e**(A + B), disregarding the negligible factor **e** *x* **e**. We note that the actual error in the result will frequently be much less than the maximum error we have indicated, for the partial errors are rarely maxima and may be compensated either partially or wholly.

EDITORIAL

The editors of the American Contract Bridge League's Official Encyclopaedia of Bridge state that the *Theorié Mathematique du Bridge* by Borel & Chéron made a major contribution to the technical development of the game. This is certainly true, but the book is not merely linked to bridge and bridge mathematics. It is one of a series of monographs on probabilities which were published under the direction of the late Mons. Émile Borel.

When the second edition was published in 1955 that series consisted of eight volumes, of which this was the fifth.

It has been a privilege to be associated with the work of two such eminent authors, and I trust that this translation will not be found inadequate.

Originally the translation was made for my own benefit but I feel that this work should be available to those who do not read French. It is a translation, so there has been practically no revision of the text – the only alteration of substance being to take into account the premium of 50 points for making a doubled or redoubled contract (section **59C**). The French text has been followed so closely that there is no section **83** in the translation, that section not appearing in the original.

It is obvious that a work of this nature is scarcely a commercial proposition today. To avoid a lengthy round of prospective publishers I approached Alan Truscott and after some correspondence suggested to him that we ask C. C. Wei to act as godfather to this translation.

Mr. Wei is known for his altruistic interest in bridge and for inventing the Precision Club bidding system. While working on that system he found Borel & Chéron very useful. He agreed to arrange publication, thus performing one more service for the bridge world.

In order to keep costs to a minimum I suggested that publication take place in Taiwan. For reasons of economy it was agreed that my typescript be offset. In the meantime numerous cross-references had been inserted in ink. To retype this lengthy work would have meant considerable delay, so we decided to overlook this rather amateurish effect.

In the original some passages were printed in italics. As my typewriter does not have this print I have underlined.

If readers derive only a fraction of the pleasure and instruction I have received from this work my efforts will be amply rewarded.

I wish to thank Émile Borel's heiress, Mme. André Lange-Appell and Mons. André Chéron for agreeing to this publication.

Finally, I would like to thank Mr. Wei for publishing this work and Alan Truscott for acting as intermediary.

CAPE TOWN. February, 1974. ALEC TRAUB.

INTRODUCTION.

Plan of this work: Our plan is very simple; we have applied successively the calculus of probabilities to various phases of the game of Bridge, as they occur in chronological order.

First of all the cards are shuffled. We have studied the problem of the shuffle from points of view both theoretical and practical in a manner much more serious and profound than has been done up to the present; this is the content of Chapter I.

Next the cards are dealt to the four players; Chapter II is devoted to the probability of the different categories of distribution of the cards after the deal.

Once the cards are distributed, each player picks up his hand and we start the auction; the purpose of Chapter III is to show how the objective probabilities calculated in Chapter II are modified by the fact that each player knows exactly what his own hand contains and also by the more or less exact information he gets from the bids of the other players. The probabilities become subjective, that is, they are not the same for all four players, for each of them sees only his own hand; further, if we assume that, the hands having been dealt, Paul gives up his place to Peter, the probabilities are not necessarily the same for both of them, for not all players interpret the same bid in exactly the same way. This phase of the game continues until the moment when the first card is played, that is to say it includes theopening lead.

The fourth phase is the play of the cards; dummy is faced, so that each player now knows at least two hands, or twenty-six cards; the subjective probabilities are modified anew; they are, further, modified during the course of play by the information given by the cards played. This is the content of Chapter IV.

Then the fifth phase; we enter the score. We study, in Chapter V, the manner in which the rules of scoring permit us, in accordance with the probabilities calculated in the preceding Chapters, to calculate the mathematical expectancy of each player in different eventualities. The best line of play is that which gives the greatest mathematical expectancy. Most of the Notes are devoted mainly to developments of more theoretical nature, specially intended for those players who are at the same time mathematicians. Those who are interested in the calculus of probabilities in itself will also note how the science of probability can derive benefit from certain questions which arise in connection with the more profound study of a game of cards.

Note VIII, in particular, is especially addressed to those of our readers who are not familiar with the calculus of probabilities. The ideas which they will acquire in reading this book will permit them to avoid certain errors which are frequently committed as a result of too hasty reasoning. **Note IX**, about the problems of the finesse, will, we hope, interest all bridge players.

All bridge players worthy of the name use either consciously or unconsciously the calculus of probabilities, that is, they decide their line of play according to their more or less instinctive valuation of certain probabilities. It is probable that the better players will find in our rigorous evaluations confirmation of the majority of the rules which they have adopted; but they will on occasion be surprised to find that the calculus does not agree with their intuition, and so they will be led to correct the latter, to their profit.

In order to find his way about a strange town a traveller can obtain complete theoretical knowledge of it by studying its plan carefully while still on the train or on board the boat. However, more often the traveller will consider it simpler and pleasanter not to study the map until he has moved freely about the town and seen certain of the streets or areas in which he is most interested;

then the plan becomes for him a living thing and not simply a confused abstraction of lines and words. Certain travellers combine both methods, making first of all a quick study of the map in its broad outlines and examining it afterwards in a more detailed manner when they return from each of their walks. We recommend this latter method for those players who wish to get the greatest benefit from this book; in reading quickly through it, first of all, their attention will be drawn to certain facets of the game; when they play afterwards they will themselves remark on those probability problems which they meet most frequently during the course of the game and, referring to the appropriate passages of the book, they will find the simplest method of obtaining the solutions.

Advice to certain readers.

Those readers who are not familiar with the calculus of probabilities will be interested, before settling down to the study of this work, to learn fundamental ideas of factorials, combinations, probabilities (section **13**), and of permutations (section **16**). They will then study Bayes' Theorem. After this preliminary initiation they will resume their study of the book in accordance with the above advice.

	♠ –	
	♡ –	
	♢Q-8-7-6-5-4-3-2	
	♣A-Q-10-8-4	
♠A-K-Q-J		♠6-5-4-3-2
♡A-K-Q-J		♡10-9-8-7-2
♢A-K		♢J-10-9
♣K-J-9		♣ –
	♠10-9-8-7	
	♡6-5-4-3	
	♢ –	
	♣7-6-5-3-2	

Bond North	Meyer East	M. South	Drax West
7♣	Pass	Pass	X
XX			

James Bond's Grand Slam

(Duke of Cumberland Hand)

Harold Stirling (Mike) Vanderbilt (1884-1970)

I. THE SHUFFLE.

1. The practical problem.

The problem of the shuffle is the most important of those which arise when we study the mathematical theory of any game; in effect it controls all the applications of the calculus of probability to the theory of games, for these applications are based on the hypothesis that the probabilities of the various possible distributions, whose number is very large (see Chapter II), are all equal among themselves. If this hypothesis becomes invalid because of the imperfect shuffle these is the risk that the conclusions that we draw become false.

It is necessary to note that if we use a new pack, the cards are generally not arranged haphazardly and that, if we use, as is most often the case, a pack which has already been used in a previous deal, the rules of the game themselves result in the cards being collected into tricks, of which a comparatively large number are composed of four cards of the same suit. A similar effect is produced if the four players have no auction but throw in their hands after having first grouped the cards in suits.

The practical study of the shuffle of cards is a very instructive example of a problem of the *application* of the calculus of probabilities, for here we learn to distinguish between the pure theory and the applied science.

From the theoretical point of view, the problem of the shuffle of the cards has given rise to numerous works, among which we shall merely cite those of Henri Poincaré and of Jacques Hadamard. These works, which are associated with the general theory of probabilities, come to the conclusion that, under certain conditions which are more general than occur in practice, all possible distributions of a pack of 52 cards (which number 52!) tend to become equally probable as the time factor is extended indefinitely. Whatever may be the theoretical interest of such a result, it is obvious that its practical value will be nil if the time necessary to realise this nearly absolute equality exceeds the length of time which the players can devote effectively to the shuffle; if, for example, it is necessary to devote more than an hour or even more than five minutes to obtain an almost perfect shuffle. We shall therefore tackle the problem exclusively from the practical point of view. It is concerned with knowing what is the true value of a shuffle obtained by the players by the methods which they generally use and in the time which another player requires to deal with the second pack of cards.

2. Conjurors. too regular shuffles.

It is scarcely necessary to say that in every mathematical theory of a game of cards, we exclude the case where one of the players is a completely unscrupulous conjuror. It is not possible to have a mathematical theory of fraud, or more exactly, an adroit crook can nullify all the conclusions drawn from the calculus of probabilities. This is so for all the applications of the calculus of probabilities in human problems.[3]

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3 According to the 1963 Laws of Contract Bridge, while South deals one pack of cards (let us say the red), North shuffles the blue pack and then places it between himself and West. For the following deal West takes the blue cards, passes them to South to cut, then, after completing the cut, West deals the blue cards. West is entitled to shuffle the cards before the cut, but if he does not exercise this right itmeans that North and South, that is to say two partners, have shuffled and cut the blue cards. This has the result that if North and South are two cheats and conjurors they can combine in their cheating. The laws could remedy this by prescribing that in no case could the shuffler and the cutter of the same pack of cards be partners. The effect of this prohibition would be that North having shuffled the blue cards West would then cut them himself before dealing them. Thus if North and West were associated in their trickery it would not be of any benefit to them for what one would win the other would lose.

I. The Shuffle.

As an example, let us show some properties of a too regular type of shuffle. In order to define a method of shuffling a pack of $2n$ cards, let us assume these cards to be numbered: the top card being number 1, the following number 2, the one at the bottom of the pack being number $2n$. If we deal the cards they will therefore appear in the order

$$1, 2, 3. \ldots 2n.$$

Assuming we have written the number on the cards (or have noticed the correspondence between the cards and their numbers; for example, No. 1 Ace of Spades; No. 2 Seven of Clubs, &c.); we could after the shuffle examine in which order the cards appear, reading from the top of the pack downwards. If the first is the Six of Diamonds and if the Six of Diamonds was number 7, we could say that the number 7 has risen to the first place; so the numbers appear in a certain order:

$$7, 4, 12, 1, \ldots .$$

We will call a shuffling operation perfectly regular when a pack of $2n$ cards is divided into two packets, each of n cards, and these two packets are mixed in such a way that each card of one of these packets is placed between two cards of the other. Further, we will assume that this operation is performed in such a way that the top and bottom cards of the new pack are not the same as the top and bottom cards of the original pack. Using the above notation this will place the cards in the following order:

$$n + 1, 1, n + 2, 2, n + 3, 3 \ldots . 2n, n.$$

We note in any case, that the precaution we suggest would impede (by making it unprofitable and not by making it more difficult) fraud committed by a couple of cheats but not by a trio trying to swindle the fourth player. The best method would be that while North deals the blue cards East, then South and then West successively shuffle the red cards. West then places the shuffled red pack between himself and North. For the following deal North alone cuts the cards and then passes them to East to deal. In this way the pack is shuffled successively by three players and cut by the fourth. This would have the additional advantage that the cards would be shuffled more efficiently, for the shuffle would last longer and also be affected by different methods of shuffling according to the individual custom of each player.

It is interesting to examine what happens when we perform this perfectly regular shuffle a certain number of times. It is easy to see that the cards will resume their original positions after at most $2n$ operations of this type. This very easy demonstration accords with Fermat's Theorem, and the general study of the perfectly regular shuffle is made very easily by using the known properties of the residues of successive powers of 2 in relation to the modulus $2n + 1$. In this way it is easy to determine the multiples of 4 for which a small number of perfectly regular shuffles will replace the cards in their original positions. With a pack of twenty cards six operations are sufficient. With a pack of 32 cards after five operations the original positions appear, but in reverse order, that is to say the first card has become the last, the second has become the penultimate, and so on. Obviously, after a further five operations, that is to say after a total of ten operations, we revert to the original order. With a pack of fifty-two cards it requires twenty-six operations to get the reverse pack and a total of fifty-two operations to regain the original order.

Obviously very great manual dexterity is required to execute a perfect shuffle a number of times in succession. But it is sufficient to have a regular shuffle twice running to obtain singular results.

Let us take a pack of $4n$ cards, of which the first n are Spades, the next n Hearts, then n Diamonds and lastly n Clubs. If we have two perfectly regular shuffles and if, after having cut in the ordinary way, we deal the cards to four players, we will see that one of the players has all the Spades, another all the Hearts, the third all the Diamonds,and, of course, the fourth all the Clubs. If we assume that one billion men on earth spend all their time dealing cards and each manages to complete more than a thousand deals a day it would take billions of centuries before we could hope to get such a result with a pack which was not faked.

After a single perfect shuffle with a pack prepared as we have mentioned above, (***n*** Spades, ***n*** Hearts, ***n*** Diamonds and then ***n*** Clubs), the two players of a single partnership, sitting opposite each other, will share all the Spades and Hearts, the other pair having all the Diamonds and Clubs. This combination is rather less improbable than the preceding one, but is sufficiently improbable to assert, if it happens, that the cards are faked.[4]

These results are interesting not only because they are curious, for if it is difficult for a player who has not got exceptional manual dexterity to execute a perfect shuffle; it is less improbable to suppose that certain players, when they shuffle the cards, sometimes obtain a partially regular shuffle, by which we mean that two groups of a certain number of cards are inserted one into the other in such a way that each card of one of the groups is placed between two cards of the other group. We revert to this point a little later.

In the meantime let us note that if, in accordance with the rules of the game, the cards are regularly cut before the deal, the cut results in permuting the cards between the players, that is to say has the same effect as if, after the deal, we turned the four hands round the table, for example each player picking up the cards of the player on his right instead of those in front of him. If the cut is always made, the player who shuffles the cards cannot have any interest in faking the shuffle, for example by having a perfectly regular shuffle, for he cannot know whether the exceptional hand he so prepares will be dealt to him or to one of his opponents.

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4 See Émile Borel "Practical Value and Philosophy of Probabilities". The probability of trickery or faking, if such a result is observed, is very much greater than the probability that such a result has been obtained by the natural chance of the game.

3. Definition of the elementary operation of shuffling.

Leaving now these singular and curious shuffles, we will define in a precise manner the way in which a player shuffles the cards. Most often he repeats a number of times an operation which we will call the elementary operation and which can be roughly described in the same way, although all such operations are not exactly the same when we examine them in detail.

Every elementary shuffling operation consists in dividing the original pack of cards into a certain number of packs (which need not contain more than one card) and in modifying the order of these packs. If there are only two packs, we have the operation called the cut which, as we have seen, simply produces a permutation between the four players: this is not a true shuffle.

Excluding the cut, the most simple elementary shuffling operation consists in dividing the pack into three packs which we will call A, B, C, and in placing one of these packs between the other two, resulting in placing them either in the order B, A, C, or in the order A, C, B. It is easy to see that this operation, which is simple, is most imperfect, that is to say, must be repeated a large number of times to obtain even a very average mixing of the cards (see infra Section **7**, Experimental Verification). A case of particular interest from the theoretical point of view is that where the pack A or C which we place between the other two only consists of a single card. The elementary operation then consists in taking alternately the upper and the lower cards of the pack and in placing them in the middle pack, preferably towards the middle. This simple procedure leads to a theoretical study which is interesting to develop, but which would side-track us for too long a time. Let us return to the elementary operation generally used by players. We will restrict ourselves to describing two.[5]

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5 We will describe a third one in Note I. Further Theory on the Shuffle.

I. The Shuffle.

Elementary operation A, which will be our main study, is sufficiently similar to the perfectly regular operation; we can say that it resembles the operation performed by a player who is comparatively clumsy. He divides the original pack into two packs approximately equal and inserts these two packs into each other, attempting to divide each of them into the greatest possible number of small packs[6], (certain among them having only one card, others having two, three, or even four or five). With a pack of 52 cards we soon have little difficulty, without any particular application of habit, in dividing each of the big packs into ten to fifteen small packs which contain on average from two to three cards. We note that this operation enables us to reach a perfectly satisfactory shuffle very quickly.

Certain players do not use method A, perhaps because they think there is a risk of ruining the cards. They prefer elementary operation B which consists of taking the cards in the right hand and detaching the upper pack a_1, which slips into the left hand, then pack a_2 which is placed on top of a_1, then pack a_3 which is placed below a_1, then a_4, above a_2, a_5 below a_3 and so on.

If we have separated an even number of packs, for example six, before the shuffle they will appear in the order

$$a_1, a_2, a_3, a_4, a_5, a_6,$$

and after the shuffle they will be in the order

$$a_6, a_4, a_2, a_1, a_3, a_5.$$

If we have an uneven number of packs they will change from the order

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7,$$

to the order

6 It is recommended if we repeat this operation a number of times to take care that, at least from time to time, the top card of the pack prior to the shuffle be not the top card after the shuffle. The same applies to the bottom card.

$$a_6, a_4, a_2, a_1, a_3, a_5, a_7.$$

It may be noted that if each time we detach the same number of packs and if we separate them the same way (which can easily happen if two consecutive packs are not absolutely square one to the other), we reach the original arrangement after a certain number of operations. If we have six or seven packs the original order appears after six operations.

We therefore recommend to players who use method B to be careful to vary the elementary operation sufficiently, that is to say, not to detach the same number of packets each time and to vary the thickness of these packets. With these precautions we see that method B can give sufficiently satisfactory results although less quickly than method A,

4. INFLUENCE OF THE SHUFFLE ON THE DIVISION BETWEEN THE FOUR PLAYERS.

A theoretically perfect shuffle is that which gives rigorously equal probabilities to all possible permutations of the fifty-two cards in the pack, permutations of which the number is 52! (see following chapter). But, in practice, the only thing which interests the players is the division of the cards between the four players after they have been dealt one by one according to the usual rule. Such a deal leads to one of the players having the cards with the ranks 1, 5, 9, 13, 17 . . . ., that is to say equal to a multiple of four plus one (ranks in the form $4k + 1$), to another player the cards with ranks equivalent to a multiple of 4 increased by 2 (ranks $4\boldsymbol{n} + 2$), to the third player cards with the ranks $4\boldsymbol{n} + 3$ and to the last player cards with the ranks $4\boldsymbol{n}$. This means that when we want to know which player will receive a certain card it is not the rank of that card which is the determining factor but the remainder after dividing the rank of the card by four. The cards which rank 13 and 37 will be dealt to the same player, in the same way as the cards with ranks 16 and 40.

We therefore have to see how the shuffle modifies the cards' ranks from the point of view of the remainder after dividing by four. We will study successively the elementary operations which we have called A and B.

Let us first take operation A; the pack is divided into two packets which are inserted one into the other; we assume that, after this insertion, each of these packets has been divided into the same number of elementary packets[7] and also let us assume, for the sake of argument, that the upper pack doesn't remain the upper one. If, to abbreviate, we confine ourselves to the case where there are twelve elementary packets, their position before the shuffle is:

$$P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12},$$

and becomes, after the shuffle,

$$P_7, P_1, P_8, P_2, P_9, P_3, P_{10}, P_4, P_{11}, P_5, P_{12}, P_6.$$

Let us call the number of cards in P_1 a_1, those in P_2 a_2, &c. We have

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} = 52.$$

If before the shuffle we call the rank of a card in P_7 $\boldsymbol{x}7$ and after the deal the rank of the card y_7 we have

$$y_7 = \boldsymbol{x}_7 - a_1 - a_2 - a_3 - a_4 - a_5 - a_6.$$

In the same way $\boldsymbol{x}_1$ and y_1, which designate the ranks of a card in pack P_1 before and after the shuffle give us

$$y_1 = \boldsymbol{x}_1 + a_7.$$

We will also find quite easily

7 The study will be to all the intents the same if one of the two big packs includes one elementary pack more than the other big pack.

$y_8 = \boldsymbol{x}_8 - a_2 - a_3 - a_4 - a_5 - a_6,$

$y_2 = \boldsymbol{x}_2 + a_7 + a_8,$

$y_9 = \boldsymbol{x}_9 - a_3 - a_4 - a_5 - a_6,$

$y_3 = \boldsymbol{x}_3 + a_7 + a_8 + a_9,$

$y_{10} = \boldsymbol{x}_{10} - a_4 - a_5 - a_6,$

$y_4 = \boldsymbol{x}_4 + a_7 + a_8 + a_9 + a_{10},$

$y_{11} = \boldsymbol{x}_{11} - a_5 - a_6,$

$y_5 = \boldsymbol{x}_5 + a_7 + a_8 + a_9 + a_{10} + a_{11},$

$y_{12} = \boldsymbol{x}_{12} - a_6,$

$y_6 = \boldsymbol{x}_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12}.$

We see that the y's are derived from the ***x***'s, either by the addition of the following numbers

$s_1 = a_7,$

$s_2 = a_7 + a_8,$

$s_3 = a_7 + a_8 + a_9,$

$s_4 = a_7 + a_8 + a_9 + a_{10},$

$s_5 = a_7 + a_8 + a_9 + a_{10} + a_{11},$

$s_6 = a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12}$

or by subtracting the following numbers:

$s_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$s_8 = a_2 + a_3 + a_4 + a_5 + a_6$

$s_9 = a_3 + a_4 + a_5 + a_6$

$s_{10} = a_4 + a_5 + a_6$

$s_{11} = a_5 + a_6$

$s_{12} = a_6$

The numbers $s_1, s_2, \ldots . s_{12}$ are connected by the sole relation $s_6 + s_7 = 52$, a relationship which corresponds with the fact that the packs P_6 and P_7 are not in reality separated by the shuffle as we have described it; they will be rejoined in their original position by the cut; everything happens as if there were only eleven packs, P_6 and P_7 forming a single one.

Let us consider two cards which belong to two different packs among these eleven; for example one in P_2 and one in P_{11}. Let $\boldsymbol{x}_2$ and $\boldsymbol{x}_{11}$ be their ranks before the shuffle; we know that according to whether $\boldsymbol{x}_{11} - \boldsymbol{x}_2$ is a multiple of four or a multiple of four augmented by 1, 2, or 3, if a card $\boldsymbol{x}_2$ is dealt to South, the second card will be dealt respectively to South, to West, to North or to East.

What happens to it after the shuffle? We will have

$$y_{11} - y_2 = \boldsymbol{x}_{11} - \boldsymbol{x}_2 - s_2 - s_{11}.$$

Supposing for the sake of argument that $\boldsymbol{x}_{11} - \boldsymbol{x}_2$ is a multiple of four, or that, before the shuffle, the two cards would have been dealt to the same player, for example South. After the shuffle, if the card y_2 comes to South, the card y_{11} will go to South, West. North, or East according to whether $s_2 + s_{11} = a_5 + a_6 + a_7 + a_8$ will be a multiple of four, or a multiple of four augmented by 3, by 2 or by 1. It is evident that as the packs are formed haphazardly the probabilities of these four eventualities are equal.[8]

So the fact that the two cards would go to the same player before the shuffle has no influence on where they will go after the shuffle, either to the same player, or to two players separated by any interval (we call the interval between South and West 1, between South and North 2, and between South and East 3).

It is understood that this reasoning does not apply to two cards in the same packet, and we must complete this first study by examining more closely the behaviour of neighbouring cards, cards which can belong to the same packet; this we will do later on and we will see that a more prolonged shuffle is necessary to disperse these cards.Nevertheless we can draw some interesting conclusions now from the results already obtained. Before mentioning these conclusions let us examine rapidly the case where we use the elementary shuffling operation which we have called B. Let us recall that if we divide the pack into seven packets

$$P_1, P_2, P_3, P_4, P_5, P_6, P_7$$

it consists in placing these in the order

$$P_6, P_4, P_2, P_1, P_3, P_5, P_7.$$

If we keep the same notation we will have

$$y_6 = \boldsymbol{x}_6 - a_1 - a_2 - a_3 - a_4 - a_5,$$

$$y_4 = \boldsymbol{x}_4 + a_6 - a_1 - a_2 - a_3,$$

$$y_2 = \boldsymbol{x}_2 + a_4 + a_6 - a_1,$$

$$y_1 = \boldsymbol{x}_1 + a_2 + a_4 + a_6,$$

$$y_3 = \boldsymbol{x}_3 + a_4 + a_6,$$

8 See Note I. Further Theory on the Shuffle for a more complete mathematical study on this point.

$y_5 = \boldsymbol{x}_5 + a_6$,

$y_7 = \boldsymbol{x}_7$

and we see that we reach the same conclusions as for Operation A.

5. Effects of the simple operation.

We have seen that Simple Operation A, in the case where the pack is divided into twelve packets (which reduces to eleven since P_6 and P_7 are not separated) has the effect of submitting to the laws of chance the interval which separates two cards which belong to two different packets. How many separate pairs of two cards are there? The number is

$$\frac{52 \times 51}{2} = 1326.$$

On the other hand, how many pairs of cards are there which belong to the same packet? For the sake of argument let us assume that among the eleven packets there is one of seven cards, two of six cards, three of five cards, three of four cards and two of three cards. Then we have

$$7+6+6+5+5+5+4+4+4+3+3 = 52.$$

In a packet of three cards there are three pairs of two cards, in a packet of four there are six, in one of five cards there are ten, in one of six cards there are fifteen and in one of seven cards twenty-one, that is in total

$$21+15+15+10+10+10+6+6+6+3+3 = 105,$$

that is to say among the cards which belong to the same packets there are 105 pairs. Since there are in total 1326 pairs, there are 1326 - 105 = 1221 pairs which belong to different packets. For these latter pairs, which are by far the more numerous, a single simple operation such as we have described has the effect of submitting the liaison which exists between them to the laws of chance. In the pack before the shuffle the cards of one of these pairs are separated by such an interval that, when we know the position of the first after the deal we could deduce the position of the second; after the shuffle each of the four possible positions has become equally probable for the second card once we know the position of the first.

We can take another point of view and consider a specified card, for example the Ace of Spades. If this card is in a packet of five cards its position in relation to the other four cards in that packet is not altered, but generally its position in relation to the other forty-seven cards is altered, for it has been submitted to chance; on average it will be altered three times out of four in relation to each of such other cards. If, for example, the deal of the original pack would have resulted in seven Spades in the hand with the Ace of Spades[9] or, on the contrary, of there being no Spades, there is no longer any reason for these rather singular circumstances to exist if the cards are dealt after the simple operation of Shuffle A. This single operation is quite sufficient to alter the pack completely and validate the application of the calculus of probabilities to any general problems of distribution which may arise. For example, all the results of Tables **3** et seq. may be applied.

We will see that it is not at all the same when we consider the subjective point of view, that is to say the point of view of a player who knows perfectly the respective positions of all the cards before the shuffle. This extreme case is scarcely likely to arise; but we can see that a player with an extraordinary memory can recall the tricks of the penultimate deal, both the position of these tricks in front of the players who collected them and then in the whole of the pack as it was reassembled at the end of the deal;

9 This is what will happen if, for example, a player who has seven Spades headed by the A-K-Q plays Spades seven times from his own hand and if these seven tricks are placed in order when the cards are collected.

this player would know almost exactly the position of the cards in the pack before the shuffle; in relation to such a player, reducing the shuffle to a single simple operation is completely insufficient and we will see later how many simple operations are required to obtain a proper result.

Reverting to simple operation A such as we have described, with twelve packets reducing to eleven, and assuming that a player knows exactly the position of the cards in the pack before the shuffle; for each card he knows which card preceded and which card follows it (we can consider the last card in the pack as preceding the first, for it would precede it in the case of a cut). If, for simplicity, we assume that each of the eleven packets contains more than one card, that is at least two, there will be twenty-two cards which are either the first or the last card of a packet, and consequently thirty cards which are in the interior of the packets. Therefore, if the player with the extraordinary memory finds in his hand the Six of Hearts and he recalls that in the original pack the Six of Hearts was preceded by the King of Spades and followed by the Queen of Hearts, he could assume without too much risk that the King of Spades is on his right (with East if he is South) and the Queen of Hearts on his left (that is to say with West). In effect, there are thirty chances in fifty-two that the Six of Hearts was an interior card in a packet and, consequently, that the two forecasts will be correct. If each of the players makes the same observation, thirty times out of fifty-two their forecasts will be correct, for the cards held by East and by West, in such a way that between the four of them they will place sixty cards (it is understood that each card will be placed at the same time by two players; if it is with East it can be placed by South or by North). Further, for the twenty-two cards which are the first or the last in a packet, one of the two forecasts will be correct (eleven on the right and eleven on the left), which makes in total eighty-two correct forecasts and only twenty-two errors. If we revert to the case of a single player who sees only his own hand and not that of dummy he therefore has approximately eighty chances in a hundred of not making a mistake in so placing the cards according to his recollection. This proportion will be reduced by a quarter, but will still be sufficiently high, if the recollection of the player is faulty in recalling the thirteen tricks as they were played in the penultimate deal (with the order in which the cards were played and naturally assuming that this order was not modified when the tricks were collected). Such an assumption is far from being unlikely.

We see therefore that the rudimentary shuffle reduced to a single simple operation A, although sufficient to permit the objective application of the results of the calculus of probabilities, is far from, being so when we consider the subjective point of view of a player endowed with excellent memory.[10]

We could speak of the study that has been made in section 4 as a macroscopic study of a pack of cards, where we have considered the pack as a whole, without paying attention to the details; now we make the microscopic study, paying attention to the smallest details – groups of two consecutive cards.

6. Breaking the sequences.

We will call the group formed by two consecutive cards of the pack as it is before the shuffle a sequence.[11] Just as, from the point of view of distribution, a pack is no different from the pack which results from the process of cutting, the last and the first cards of the pack can be regarded as forming a sequence; in this way there

10 In Note I. Further Theory on the Shuffle we will consider the general fact that all players, without requiring any memory, know that a number of tricks are composed of cards of the same suit. It would not be of interest to study this fact from the practical point of view unless we confined ourselves to the rudimentary shuffle and this is never the case with serious players.

11 It is necessary to observe that the term sequence has a totally different meaning from the term sequence used by Poker players. Here it means two cards in juxtaposition, without reference to any resemblance they may have to each other, e.g., the Two of Diamonds and the Five of Hearts.

are altogether 52 sequences. The question which we are going to study is to know how many of these sequences will be broken by the shuffle. We confine ourselves to the simple operation which we have called A. and we assume that this operation is repeated a certain number of times in succession by the player who shuffles the cards.

We have already observed that Operation A consists in separating the pack into two packets approximately equal and in subsequently dividing each of these packets into a sufficiently large number of partial packets, and in inserting the latter into each other.

According to the way in which the two packets are inserted into each other both the first and the last cards of the pack can retain their respective places, or only one of them can retain its place, or neither of them. Of these four eventualities, the last is preferable from many points of view[12]; it is to be recommended that if the player does not always observe this, it should at any rate be observed on occasions during a single shuffle.

Let us note that in the first and the last cases the two original packets are divided into the same number of partial packets, in such a way that the total number of partial packets is an even number $2\boldsymbol{a}$, while in the two cases where only one of the two cards (the first or the last) retains its place, one of the two packets contains a partial packet more than the other; the total number of partial packets is an uneven number $2\boldsymbol{a} + 1$.

We have already seen, in the example considered above, that in the case where the number of packets is even and equal to $2\boldsymbol{a}$, the number of broken sequences is $2\boldsymbol{a} - 1$, for two of the packets are consecutive; while where the number of packets is uneven, we can easily see that the number of broken sequences is equal to the

12 See Note I. Further Theory on the Shuffle.

number $2\boldsymbol{a} + 1$ of the packets. The minimum number of broken sequences is therefore at least equal to three.[12]

When we effect several simple operations successively, the number $\boldsymbol{a}$ is not exactly the same; if a player has a technique which is sufficiently regular this number $\boldsymbol{a}$ will generally deviate very little from a certain average value and it is this average value which characterises the shuffling habits of each player. From hence forth it is this average value which we designate by $\boldsymbol{a}$; the number of packets will be, whether $2\boldsymbol{a}$ or $2\boldsymbol{a} + 1$, such numbers either slightly greater or slightly less, and the number of sequences broken by each operation can, even although it is always uneven, be regarded as equal and averaging to $2\boldsymbol{a}$.

We will study the case where the number $\boldsymbol{a}$, the average number of partial packets into which each of the two packets is divided, is equal to 13; this value gives us particularly easy calculations and subsequently we shall be content to indicate rapidly the results for some other values of $\boldsymbol{a}$.

We will therefore assume that when we insert the two packets into each other, each divides into thirteen partial packets (or one into thirteen and the other into fourteen), this number varying a little, either being greater or smaller.

Under these conditions the number of sequences broken by the first simple shuffling operation will be *on average* 26, that is to say half the number of sequences.

Can we assume that two successive operations will be enough to disrupt all the sequences? This would be a grave error, for during the second operation quite often the cards will be inserted in the intervals corresponding to the sequences already broken. Let us specify – and this is what interests us exclusively – that these are the sequences which existed in the original pack, before any shuffle; the sequences which were formed after the first shuffling operation, in place of the broken sequences, do not interest us and we are completely indifferent whether they in their turn are broken.

Our argument should be as follows. The first operation **A** broke twenty-six sequences out of fifty-two, that is half; twenty-six remained; the second operation will break half the sequences of the new pack and consequently, *on average,* half of the twenty-six sequences which remained, therefore approximately thirteen sequences will remain after the second operation. A third operation **A** will break on average a further half of these thirteen, in such a way that approximately seven will remain. In the same way on average four will remain after the fourth operation **A**, two after the fifth, one after the sixth. Everyone who has had practice with probabilities knows further that, *when numbers become small, results are often very different from the average*; this average is only of significance when we repeat the same experiment a large number of times, which is precisely the case with bridge players who shuffle the cards more than a thousand times a month if they play a few hours each day.

If therefore we suggest as the object of the shuffle the breaking of all sequences, this object will be achieved more or less completely on average after six simple operations **A** (each consisting of approximately two lots of thirteen partial packets); it is almost certain that all sequences will be broken if instead of six operations we have a few more, for example ten.

We continue this discussion with the following rule: if the number of partial packets in each packet is thirteen the number of simple operations must be from six to ten. *Less than six will give an imperfect shuffle; more than ten will be wasted effort*.

With an average of ten packets instead of thirteen, nine to fourteen operations are required. With seven packets we require twelve to twenty. If we have eighteen packets, it suffices to have from four to seven.

7. Experimental verification.

The preceding theoretical considerations, which will be completed in **Note I**, do not entitle us to omit an experimental verification. The individual methods of shuffling the cards are in effect too diverse for us to hope to establish a general theory; certain players have peculiarities which a theory cannot forecast. On the other hand, in practice the different methods of shuffling show themselves in forms sufficiently different according to whether we use exclusively cards which are almost new, or whether we use a pack sufficiently long for the cards to be indistinguishable when they are dealt at the table.

It is understood, if a player confirms by one or two experiments that he is able to carry out a satisfactory operation **A** as we have described it, having care particularly not to keep in their original places either the first or the last cards, our theoretical conclusions will be correct, as has been demonstrated elsewhere by experiment.

This experimental verification consists in taking a pack in which the cards are placed in a known order and one easy to retain; we shuffle the pack and afterwards examine how the disposition of the cards has been modified. As we will see precisely in **Note I** it is sufficient to pay attention to the packets or sequences which are unchanged.

Suppose that we have arranged the cards in the following order, reading from the top, that is to say in the order in which the cards will be dealt; first all the Spades, in descending order, that is to say A, K, Q, down to the 2; then the Hearts in the same order, then Diamonds and last Clubs. We therefore know that the Seven of Diamonds is followed by the Six of Diamonds, that the Two of Spades is followed by the Ace of Hearts and the last card is the Two of Clubs followed by the Ace of Spades, which is the first. We therefore know the fifty-two sequences which, except for the

Two's followed by the Aces, are precisely natural sequences of two cards of the same suit.

We ask a player to shuffle this pack according to his usual method and, when he has finished, we face the pack on the table, without modifying the order of the cards, in such a way that we can see all of them (or alternatively we turn the cards one by one); we pay attention to the sequences which remain intact; if, for example, we see a sequence in order of the King, the Queen, the Jack and the Ten of Diamonds we note a sequence of four cards; if we see the Two of Spades followed by the Ace and King of Hearts we note a sequence of three cards. For preference we call sequences of more than four cards *packets*.

If there are several packets, or even if there is a single large packet (of seven or eight cards at least), the technique employed for the shuffle is defective, and we should ask the player to modify his technique.

If there are no packets, but only a certain number of sequences of two or three cards, exceptionally of four, the technique employed is not bad, but the operation has not been sufficiently lengthy; we should ask the player to shuffle the cards for twice the length of time (or at least for 50% longer); we could then recommence the experimental verification under the new conditions.

Finally, if there are only a small number of sequences of two cards, exceptionally one of three cards, the shuffle can be regarded as satisfactory in practice; it will be perfect if *on average* only a single sequence of two cards remains.[13]

If we count the number S of sequences broken by each elementary operation and the number N of the elementary operations, we will generally confirm that the shuffle is good

13 See Note I. Further Theory on the Shuffle.

when the product ***NS*** reaches or exceeds 150 and excellent when the product exceeds 200. If this does not happen it is proof that the technique used is sullied with a systematic fault which we should try to discover. This will be the case, for example, if the seven or eight first or last cards of the pack are never mixed.

There are certainly some players who consider the time and effort spent on the shuffle as time lost and useless labour. They perform rather vaguely certain ritual gestures and consider that they have subjected the pack to the laws of chance when in fact they have merely completed a very imperfect shuffle. If all the players at a table adopt this method, they should not be surprised if certain of the facts they observe are not always in accordance with the laws of probability; things will happen as if they were playing with a crooked roulette wheel or with loaded dice. We will examine rapidly what will be the principal consequences of an imperfect shuffle.[14]

8. Consequences of an imperfect shuffle.

We will identify the imperfection of a shuffle by the average number of sequences which remain, that is to say which have not been broken. Let us however note that chance can reunite a sequence which has already been broken; we will show in **Note I** that this will happen on average once in every entire pack; if therefore on average $k + 1$ sequences remain, we must accept that, on average, k sequences have not been broken and that, on average, one broken sequence has been reunited.

Let us now assume that we have observed one sequence in the tricks of the penultimate deal; we recall that the King of Spades was covered by the Ace of Spades, in such a way that we have sequence King, Ace or K, A (superfluous to repeat Spades). Suppose that South, collecting his hand, at first only sees a single

14 We should remember that the player who deals the cards has the right to shuffle them anew if he considers the existing shuffle insufficient.

card and that card is the Ace. What can he assume about the position of the King?

We have assumed that the average number of sequences which are not broken is ***k*** (as ***k*** is an average, it is not necessarily a whole number); it will be convenient for us, instead of saying that ***k*** sequences have remained out of fifth-two, to talk of the number of sequences which remain as ***x***% $\frac{x}{100} = \frac{k}{52}$; we could equally say, in using the language of probabilities, that the probability that a certain sequence will remain, for example, the sequence K, A, which interests us, is ***x***% and consequently the probability that it will not remain is (l00 - ***x***)%.

We can therefore reason as follows: in ***x*** cases out of 100 the sequence has been retained and the King is certainly with East; in 100 - ***x*** cases in a hundred the sequence was not retained and the King has equal chances of being in one of the 51 places which are not filled by the Ace. Out of these 51 places, there are thirteen which are with East[15], thirteen with North, thirteen with West and twelve with South. The probabilities that the King will be with East, North, West and South are therefore respectively:

East . . . $x + \frac{13}{51}(100 - x)\%$

North . . . . . $\frac{13}{51}(100 - x)\%$

West . . . . . . $\frac{13}{51}(100 - x)\%$

South . . . . . $\frac{12}{51}(100 - x)\%$

and we confirm immediately that the sum of these probabilities amounts to 100%.

15 Among the thirteen places which the King could occupy with East there is the place which it occupies above the Ace where the sequence has been reunited. The probability of this eventuality is $\frac{1}{51}$, as that of each of the others.

Let us assume now that South has collected his hand and exposed North's (Dummy's). If he sees the King in his own or North's hand the problem which he has posed has been resolved and is no longer of interest. In fact it is only after having confirmed that neither South nor North has the King that there is any problem. He therefore might be tempted to reason as follows: if the sequence has not been broken, the King is surely with East; this will happen in x out of 100 cases; if it has been broken, the chances are equal for both East and West; this amounts to $100 - x$ cases out of 100. The probabilities are therefore

$$\text{East} \quad x + \frac{1}{2}(100 - x)\%.$$

$$\text{West} \quad \frac{1}{2}(100 - x)\%.$$

This method of reasoning is grossly inaccurate; it is the first example we encounter of false reasoning, committed because we have forgotten that *chance takes a hand only at the moment the 52 cards are dealt to the four players*; we have not the right to neglect the original chance and substitute summary reasoning for it; at the very least, it cannot be done except with the greatest care and in certain cases which have been studied meticulously.

Here is the correct reasoning: we have found, where the only card we know is the Ace in South's hand, the probabilities for the King to be with East, North, West and South. When we find that it is neither with South nor North *the relationship between the probabilities of East and West have not been modified*, and the sum of these probabilities must be equal to unity (100 in percentages). During the remainder of this book we will call this *the method of the rule of three*.

By multiplying by 51 the probabilities obtained for East and West they become:

$$\text{East} \quad 51x + 13(100 - x) = 1\,300 + 38x.$$

$$\text{West} \quad 13(100 - x) = 1\,300 - 13x.$$

So that the relationship between these probabilities is not modified and so that their sum becomes equal to 1 we must take:

$$\text{East} \ldots\ldots \quad \frac{1300+38x}{2600+25x},$$

$$\text{West} \ldots\ldots \quad \frac{1300-13x}{2600+25x};$$

or, in percentages:

(A)

$$\text{(East} \ldots\ldots \quad \frac{1300+38x}{26+0{,}25x}\ \%$$

$$\text{(West} \ldots\ldots \quad \frac{1300-13x}{26+0{,}25}\ \%$$

The difference between these two probabilities (in percentages) correctly calculated is, when x is small, very close to $2x$, while the difference of the probabilities which we obtain from incorrect reasoning is only equal to x. We see that the difference is considerable, for, with correct reasoning, the advantage which South obtains from his knowledge of the possible sequence KA is almost double that which it is with incorrect reasoning.

Let us show briefly by an abbreviated and approximate calculation the profound reason why the original argument is incorrect. Let us assume that x is equal to 20, that is that the probability that the sequence will not have been broken is equal to 20%. Consequently, eighty times out of a hundred the sequence will have been broken and we agree that the probabilities are therefore equal that the King will be with East, North, West or South (in reality the probability is slightly less for South because he has only twelve unknown cards while there are thirteen unknown for each of the other three players; our approximation consists in ignoring the difference between $\frac{12}{51}$ and $\frac{13}{51}$. It is a small enough error.

GENERAL INTEREST

134 PROBABILITY TABLES, THEIR USES, SIMPLE FORMULAS, APPLICATIONS & 4000 PROBABILITIES

Originally published in 1940, and revised in 1954, this classic work on mathematics and probability as applied to Bridge first appeared in English translation in 1974, but has been unavailable for many years. This new edition corrects numerical errors found in earlier texts; it revises the previous English translation where needed and corrects a number of textual and typographical errors in the 1974 edition. Tables have been included again in the text, as they were in the original edition. The chapter on Contract and Plafond scoring has been retained as continuing to serve its intended purpose. The chapters on shuffling, although no longer applicable to Duplicate Bridge, are included for the benefit of those interested in the mathematics of all card games. All, it is hoped, without too many new errors being introduced.

ÉMILE BOREL (1871-1956) made contributions to mathematics, it can be argued, that introduced our era of probabilistic, quantitative decision making, so adaptable to the computer and thereby so pervasive today. He published more than fifty papers on probability between 1905 and 1950. Between 1921 and 1927 he published papers on game theory and was first to define games of strategy, publishing various papers and finally, *Applications aux jeux de hazard* in 1938, of which this book formed a part.

ANDRÉ CHÉRON (1895-1980) was a top-ranked chess player of the generation of Capablanca (1888-1942); both he and Capablanca turned to bridge as a 'more interesting game' in the 1930s.

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